Generating function for Bell numbers

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Introduction

Partition of a set

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- This is a 3-class partition of [5].
- Here is a list of all 7 partitions of [4] into 2 classes:

  $\{12\}\{34\}$; $\{13\}\{24\}$; $\{14\}\{23\}$; $\{123\}\{4\}$; $\{124\}\{3\}$; $\{134\}\{2\}$; $\{1\}\{234\}$.

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- The first pile consists of all those partitions of $[n]$ into k classes in which the letter n lives in a class all by itself.

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- The first pile consists of all those partitions of $[n]$ into k classes in which the letter n lives in a class all by itself.
- The second pile consists of all other partitions in which the letter n lives in a class with other letters.

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- Now consider the second pile.
- There the letter n always lives in a class with other letters.
- Therefore, if we march through that pile and erase the letter n wherever it appears, our new pile would contain partitions of $n-1$ letters into k classes.
- However, each one of these partitions would appear not just once, but k times.

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- Hence this list contains exactly $2\left\{ {3 \atop 2} \right\}$ partitions.
- Therefore in the general case the second pile must contain $k\left\{ {n-1 \atop k} \right\}$ partitions before the erasure of n.
- It must therefore be true that

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}.$$

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- Lets try to find the generating function for the sum of face values of 1 & 2 dice.
- For a standard six-sided die, there is exactly 1 way of rolling each of the numbers from 1 to 6. Hence, we can encode this as the power series $R_1(x) = x^1 + x^2 + x^3 + x^4 + x^5 + x^6$.

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- The exponents represent the value rolled on the die, and the coefficients represent the number of ways this value can be attained.

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- For rolling 2 dice, we could likewise list out the possible sums, and arrive at
  $R_2(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$

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- **etc.**

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by x^n and sum on n to get

$$B_k(x) = xB_{k-1}(x) + kxB_k(x)$$

where $k \geq 1$ and $B_0(x) = 1$.

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- This leads to

$$B_k(x) = \frac{x}{1 - kx} B_{k-1}(x)$$

and to the formula

$$B_k(x) = \sum_n \left\{ {n \atop k} \right\} x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}, k \geq 0.$$

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- Continuing our process to find an explicit formula for Stirling numbers of the second kind, we expand the partial fraction in question

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- To find the α's, say α_r for $1 \leq r \leq k$, we multiply both sides by $1 - rx$ and put $x = 1/r$. We get

$$\begin{aligned}\alpha_r &= \frac{1}{(1-1/r)\cdots(1-(r-1)/r)(1-(r+1)/r)\cdots(1-k/r)} \\ &= (-1)^{k-r}\frac{r^{k-1}}{(r-1)!(k-r)!}.\end{aligned}$$

Generating function for Stirling Numbers

- Now, we use the formula for $B_k(x)$ and α_r to get an explicit formula for $\left\{ {n \atop k} \right\}$ where $n \geq k$ (in the following process, $[x^n]$ denotes the coefficient of x^n in the expression):

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- The sequence of Bell numbers begins as $1, 1, 2, 5, 15, 52, \ldots$.
- Can we find an explicit formula for the Bell numbers, $b(n)$?
- Yes, we can ! If we sum the formula of Stirling number from $k = 1$ to n we will have an explicit formula for $b(n)$.

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- Thus the result is that

$$b(n) = \sum_{k=1}^{M} \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{(r)!(k-r)!}$$
$$= \sum_{r=1}^{M} \frac{r^{n-1}}{(r-1)!} \left\{ \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(k-r)!} \right\}.$$

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- But now the number M is arbitrary, except that $M \geq n$. Since the partial sum of the exponential series in the curly braces above is so inviting, let's keep n and r fixed, and let $M \longrightarrow \infty$.

Calculating $b(n)$

- This gives the following remarkable formula for the Bell numbers:

$$b(n) = \frac{1}{e} \sum_{r \geq 0} \frac{r^n}{r!}$$

for $n \geq 0$.

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- The above formula is not feasible for computation and we try to look for a generating function of the Bell numbers in the form:

$$B(x) = \sum_{n \geq 0} \frac{b(n)}{n!} x^n.$$

Calculating $b(n)$

- We find $B(x)$ explicitly by multiplying both sides of the formula by $\frac{x^n}{n!}$ and sum over all $n \geq 1$:

$$B(x) - 1 = \frac{1}{e} \sum_{n \geq 1} \frac{x^n}{n!} \sum_{r \geq 1} \frac{r^{n-1}}{(r-1)!}$$

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$$= \frac{1}{e} \{e^{e^x} - e\}$$

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&= \frac{1}{e} \sum_{r \geq 1} \frac{1}{r!} \sum_{n \geq 1} \frac{(rx)^n}{n!} \\
&= \frac{1}{e} \{ e^{e^x} - e \} \\
&= e^{e^x - 1} - 1.
\end{aligned}
$$

- So we get that the exponential generating function of the Bell numbers is $e^{e^x - 1}$ i.e., the coefficient of $x^n/n!$ in the power series expansion of $e^{e^x - 1}$ is the number of partitions of a set of n elements.

References

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