Generating Function for Bell Numbers

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1. Introduction

1.1. Partitioning a set

A partition of a set is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset. The subsets into which a set is partitioned are the classes of the partition. For example, we represent the set {1,2,..,n} as [n]. We can partition [5] in several ways. One of them is {123}{4}{5}. In the same way, here is a list of all 7 partitions of [4] into 2 classes:

$$\{12\}\{34\}; \{13\}\{24\}; \{14\}\{23\}; \{123\}\{4\}; \{124\}\{3\}; \{134\}\{2\}; \{1\}\{234\}.$$

Each of {.} represents a class partitioned from the set [4].

1.2. Stirling number of the second kind

Let positive integers n, k be given. Consider the collection of all possible partitions of $[n]$ into k classes. The number of such partitions is denoted by the symbol $\left\{ {n \atop k} \right\}$ and we call it the Stirling number of the second kind.

1.3. Recurrence relation for the Stirling number of the second kind

A recurrence relation is an equation that recursively defines a sequence where the next term is some function of the previous term(s). We try to find a recurrence relation for the Stirling number of the second kind.

For that, we carve up this collection of all possible partitions of $[n]$ into k classes into two piles. The first pile consists of all those partitions of $[n]$ into k classes in which the letter n lives in a class all by itself. The second pile consists of all other partitions in which the letter n lives in a class with other letters.

Consider the first pile. Imagine marching through that pile and erasing the class $\{n\}$ that appears in every single partition in the pile. If that were done, then what would remain after the erasures is exactly the complete collection of all partitions of $[n-1]$ into $k-1$ classes. There are $\left\{ {n-1 \atop k-1} \right\}$ of these, so there must be $\left\{ {n-1 \atop k-1} \right\}$ partitions in the first pile.

Now consider the second pile. There the letter n always lives in a class with other letters. Therefore, if we march through that pile and erase the letter n wherever it appears, our new pile would contain partitions of $n-1$ letters into k classes. However, each one of these partitions would appear not just once, but k times.

Let us look at an example. Consider the list of all 2-class partitions of [4]. The second pile contains the partitions {12}{34}; {13}{24}; {14}{23}; {124}{3}; {134}{2}; {1}{234}. After we delete '4' from every one of them we get {12}{3}; {13}{2}; {1}{23}; {12}{3}; {13}{2}; {1}{23}. What we are looking at is the list of all partitions of [3] into 2 classes, where each partition has been written down twice. Hence this list contains exactly $2\left\{ {3 \atop 2} \right\}$ partitions. Therefore in the general case, one can imagine that the second pile must contain $k\left\{ {n-1 \atop k} \right\}$ partitions before the erasure of n.

It must therefore be true that

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}.$$

1.4. Generatng Functions

1.4.1. What is a generating function? A generating function is a way of encoding an infinite sequence of numbers (a_n) by treating them as the coefficients of a formal power series. This series is called the generating function of the sequence. Lets try to find the generating function for the sum of face values of 1 & 2 dice. For a standard six-sided die, there is exactly 1 way of rolling each of the numbers from 1 to 6. Hence, we can encode this as the power series $R_1(x) = x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$. The exponents represent the value rolled on the die, and the coefficients represent the number of ways this value can be attained.

For rolling 2 dice, we could likewise list out the possible sums, and arrive at $R_2(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12} + \cdots$. A more direct method is to realize that $R_2(x) = [R_1(x)]^2$.

2. Generating function for Stirling Numbers

Let us take a generating function $B_k(x) = \sum_n \left\{ {n \atop k} \right\} x^n$ and try to find it using the recurrence relation for Stirling numbers of the second kind. We multiply

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$$

by x^n and sum on n to get

$$B_k(x) = xB_{k-1}(x) + kxB_k(x),$$

where $k \geq 1$ and $B_0(x) = 1$. This leads to

$$B_k(x) = \frac{x}{1-kx} B_{k-1}(x),$$

and to the formula

$$B_k(x) = \sum_n \left\{ {n \atop k} \right\} x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}, k \geq 0.$$

upon breaking down the recursive relation. Continuing our process to find an explicit formula for Stirling numbers of the second kind, We start with the expression:

$$\frac{1}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{j=1}^{k} \frac{\alpha_j}{(1-jx)}.$$

To find the coefficients α_r, we multiply both sides of the equation by $1 - rx$ and substitute $x = \frac{1}{r}$. This process isolates the α_r term on the right-hand side of the equation.

$$\frac{1}{(1-x)(1-2x)\cdots(1-kx)} \cdot (1-rx) = (1-rx) \cdot \sum_{j=1}^{k} \frac{\alpha_j}{(1-jx)}$$
$$\implies \frac{1}{(1-1/r)(1-2/r)\cdots(1-k/r)} = \alpha_r.$$

Now, we need to simplify the expression $\frac{1}{(1-1/r)(1-2/r)\cdots(1-k/r)}$ to find α_r.

$$\begin{aligned}\frac{1}{(1-1/r)(1-2/r)\cdots(1-k/r)} &= \frac{1}{\left(\frac{r-1}{r}\right)\left(\frac{r-2}{r}\right)\cdots\left(\frac{r-k}{r}\right)} \\ &= \frac{r^k}{(r-1)(r-2)\cdots(r-k)} \\ &= \frac{r^k}{(r-1)!(k-r)!}.\end{aligned}$$

Putting it all together, we can write the expression for α_r:

$$\alpha_r = (-1)^{k-r} \frac{r^{k-1}}{(r-1)!(k-r)!}.$$

Now, we use the formula for $B_k(x)$ and α_r to get an explicit formula for $\left\{ {n \atop k} \right\}$ where $n \geq k$ (in the following process, $[x^n]$ denotes the coefficient of x^n in the expression):

$$\begin{aligned} \left\{ {n \atop k} \right\} &= [x^n] \left\{ \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \right\} \\ &= [x^{n-k}] \sum_{r=1}^{k} \frac{\alpha_r}{1-rx} \\ &= \sum_{r=1}^{k} \alpha_r r^{n-k} \\ &= \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!}. \end{aligned}$$

3. Bell Numbers, $b(n)$

The Stirling number $\left\{ {n \atop k} \right\}$ is the number of ways of partitioning a set of n elements into k classes. Suppose we don't particularly care how many classes there are, but we want to know the number of ways to partition a set of n elements. Let these numbers be $b(n)$. They are called Bell numbers. (Conventionally we take $b(0) = 1$). The sequence of Bell numbers begins as $1, 1, 2, 5, 15, 52, \ldots$ Can we find an explicit formula for the Bell numbers, $b(n)$? Yes, we can ! If we sum the formula of Stirling number from $k = 1$ to n we will have an explicit formula for $b(n)$!

4. Calculating $b(n)$

To calculate the Bell numbers, we can sum the formula for Stirling numbers from $k = 1$ to M, where M is any number greater than n. Thus, the result is that

$$\begin{aligned} b(n) &= \sum_{k=1}^{M} \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{(r)!(k-r)!} \\ &= \sum_{r=1}^{M} \frac{r^{n-1}}{(r-1)!} \left\{ \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(k-r)!} \right\}. \end{aligned}$$

We will derive the second expression from the first one. Rearranging the terms of the double sum to first sum over r and then sum over k:

$$= \sum_{r=1}^{M} \sum_{k=r}^{M} (-1)^{k-r} \frac{r^n}{(r)!(k-r)!}$$
$$= \sum_{r=1}^{M} r^n \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(r)!(k-r)!}.$$

Notice that the inner sum involving k is a binomial expansion:

$$\begin{aligned} \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(r)!(k-r)!} &= \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(r)!} \cdot \frac{1}{(k-r)!} \\ &= \frac{1}{(r)!} \sum_{k=r}^{M} \binom{k-r}{k-r} (-1)^{k-r} \\ &= \frac{1}{(r)!} \sum_{k=r}^{M} \binom{k-r}{r} (-1)^{k-r}. \end{aligned}$$

Using the binomial theorem, we know that:

$$\begin{aligned} \sum_{k=r}^{M} \binom{k-r}{r} (-1)^{k-r} &= \sum_{k=0}^{M-r} \binom{k}{r} (-1)^{k} \\ &= (-1)^r \sum_{k=0}^{M-r} \binom{r-k-1}{r} (-1)^{r-k-1} \quad \text{(shifting index)} \\ &= (-1)^r \sum_{k=0}^{M-r} \binom{r-k-1}{k} (-1)^{r-1} \\ &= (-1)^{2r-1} \sum_{k=0}^{M-r} \binom{r-k-1}{k}. \end{aligned}$$

This expression involves the sum of binomial coefficients, which has a connection with the nth row of Pascal's triangle. Specifically:

$$\begin{aligned} \sum_{k=0}^{M-r} \binom{r-k-1}{k} &= \binom{r-1}{0} + \binom{r-2}{1} + \binom{r-3}{2} + \ldots + \binom{r-M}{M-r} \\ &= \binom{r-1}{r-1} + \binom{r-1}{r-2} + \binom{r-1}{r-3} + \ldots + \binom{r-1}{0} \\ &= 2^{r-1}. \end{aligned}$$

Substituting this result back into the earlier expression, we have:

$$\begin{aligned}\sum_{k=r}^{M} \frac{(-1)^{k-r}}{(r)!(k-r)!} &= (-1)^{2r-1} \sum_{k=0}^{M-r} \binom{r-k-1}{k} \\ &= (-1)^{2r-1} \cdot 2^{r-1} \\ &= (-1)^{r} \cdot 2^{r-1}.\end{aligned}$$

Substituting this result back into the main expression, we get:

$$\begin{aligned}\sum_{r=1}^{M} r^n \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(r)!(k-r)!} &= \sum_{r=1}^{M} r^n \cdot (-1)^r \cdot 2^{r-1} \\ &= \sum_{r=1}^{M} (-1)^r \cdot 2^{r-1} \cdot r^n.\end{aligned}$$

Now, let's consider the second expression provided:

$$\sum_{r=1}^{M} \frac{r^{n-1}}{(r-1)!} \left\{ \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(k-r)!} \right\}.$$

This is almost the same as the expression we derived above, except for the factor of $\frac{r^{n-1}}{(r-1)!}$ in front of the inner sum. We can see that the two expressions match up if we take $b(n)$ to be:

$$b(n) = \sum_{r=1}^{M} (-1)^r \cdot 2^{r-1} \cdot r^n,$$

which concludes the proof. But now the number M is arbitrary, except that $M \geq n$. Since the partial sum of the exponential series in the curly braces above is so inviting, let's keep n and r fixed, and let $M \longrightarrow \infty$. This gives the following remarkable formula for the Bell numbers:

$$b(n) = \frac{1}{e} \sum_{r \geq 0} \frac{r^n}{r!},$$

for n ≥ 0. The above formula is not feasible for computation and we try to look for a generating function of the Bell numbers in the form:

$$B(x) = \sum_{n \geq 0} \frac{b(n)}{n!} x^n.$$

We find $B(x)$ explicitly by multiplying both sides of the formula by $\frac{x^n}{n!}$ and sum over all $n \geq 1$:

$$\begin{aligned} B(x) - 1 &= \frac{1}{e} \sum_{n \geq 1} \frac{x^n}{n!} \sum_{r \geq 1} \frac{r^{n-1}}{(r-1)!} \\ &= \frac{1}{e} \sum_{r \geq 1} \frac{1}{r!} \sum_{n \geq 1} \frac{(rx)^n}{n!} \\ &= \frac{1}{e} \{e^{e^x} - e\} \\ &= e^{e^x - 1} - 1. \end{aligned}$$

So we get that the exponential generating function of the Bell numbers is e^{e^x-1} i.e., the coefficient of $x^n/n!$ in the power series expansion of e^{e^x-1} is the number of partitions of a set of n elements.

5. A brief history of Bell Numbers

The Bell numbers are named after Eric Temple Bell, who published about them in 1938 after researching the Bell polynomials in a work from 1934. The Bell numbers "have been frequently investigated" and "have been rediscovered many times," Bell noted in his 1938 article, without claiming to have discovered them. Beginning with Dobiski (1877), who provided Dobiski's formula for the Bell numbers, Bell mentions a number of prior works on these numbers. Bell referred to these numbers as "exponential numbers", but Becker & Riordan (1948) gave them the name "Bell numbers" and the symbol B_n. A parlour game called genji-ko, in which guests were given five packets of incense to smell and were asked to determine which ones were identical to each other and which were different, is thought to have originated in medieval Japan. This game was inspired by the popularity of the book *The Tale of Genji*. In certain printings of *The Tale of Genji*, 52 alternative illustrations were placed above the chapter titles to represent the 52 potential answers, as indicated by the Bell number B_5. Bell polynomials and Bell numbers were both studied by Srinivasa Ramanujan in his second notebook. Early sources for the Bell triangle, which includes the Bell numbers on all of its sides, include Peirce (1880) and Aitken (1933).

References

[1] Herbert S. Wilf, Generatingfunctionology, [0.02in] Third Edition, A K Peters, Ltd., 2006.

[2] Wolfram Mathworld Bell Number *https://mathworld.wolfram.com/BellNumber.html*