Normalized Cuts and Image Segmentation

Based on a paper by Jianbo Shi and Jitendra Malik

Linear Algebra and its Applications

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Outline

Perceptual grouping in vision

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Laws of Organization in Perceptual Forms
Max Wertheimer (1923)

Key Factors

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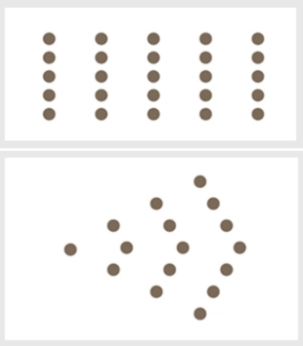

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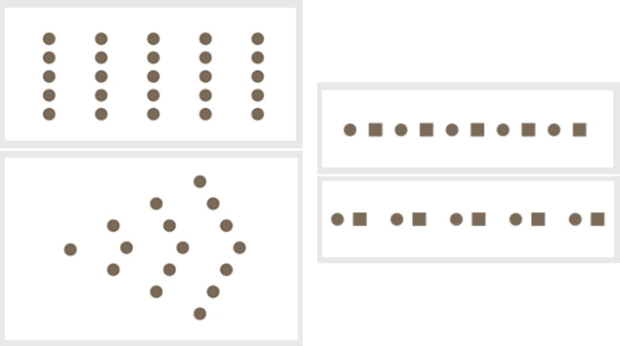

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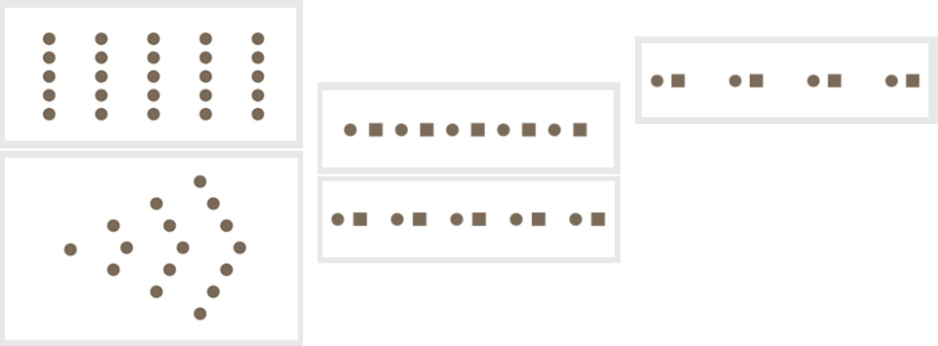

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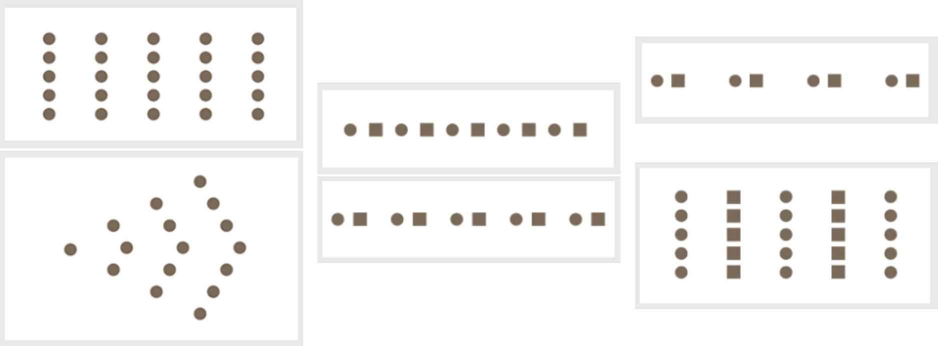

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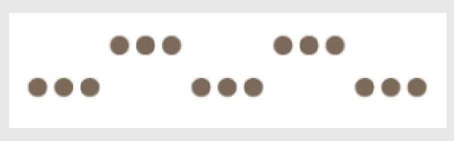

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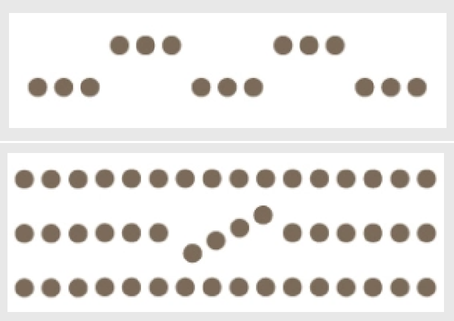

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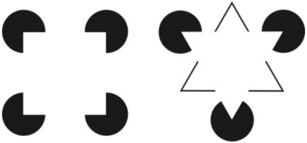

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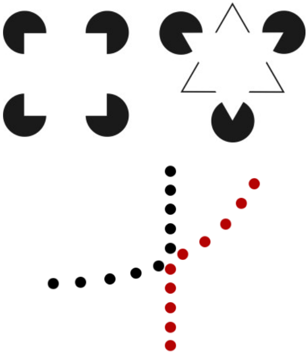

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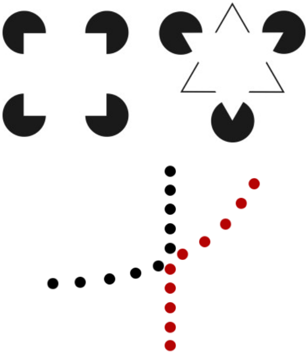

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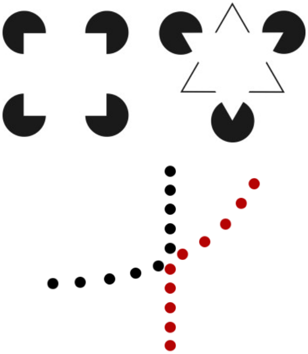
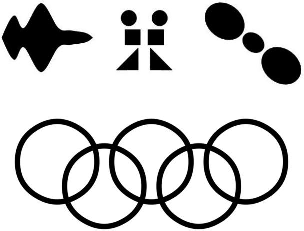
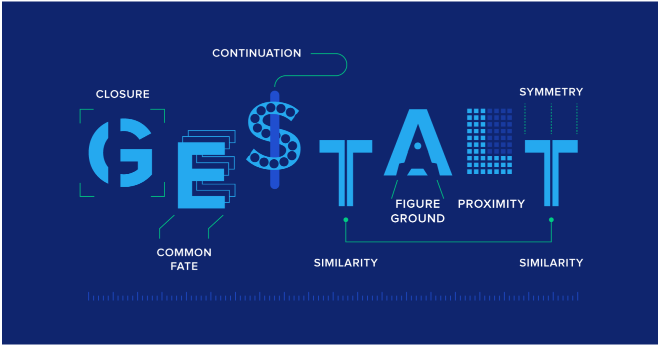




Gestalt's Law



Selecting the best partition

Two aspects to consider here -

- First aspect
  - Single correct answer may not exist.
  - Depends on prior world knowledge - Important to specify
  - Could be high level, mid level or low level
  - Low level - brightness, color, texture etc.
    High/mid level - Symmetry, object models etc.
- Second aspect
  - Hierarchy of partitioning
  - Tree structure corresponding to hierarchical partitioning better than a single flat partition

Low level knowledge - Hierarchical partitions.

High level knowledge - Confirm the segments

Graph Theoretic Formulation

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1. **What is the precise criterion for a good partition?**
2. **How can such a partition be computed efficiently?**

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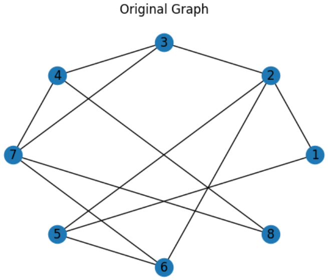
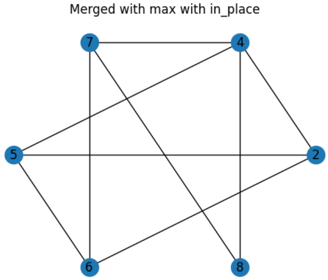
Wu and Leahy proposed a clustering approach based on MinCut

Mincut


Original Graph
3
4
2
7
1
5
8
6


Merged with max with in_place
7
4
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Mincut

Figure: Original image

Figure: Ncut

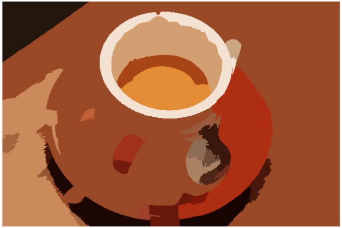
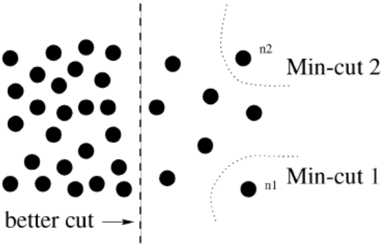


Figure: Min cut

Section

Normalized Cut



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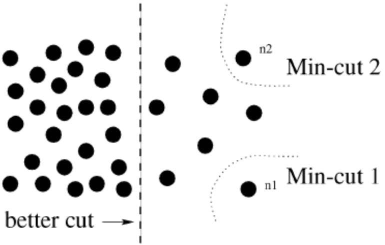

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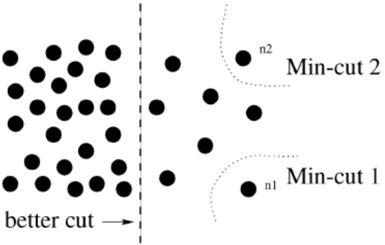

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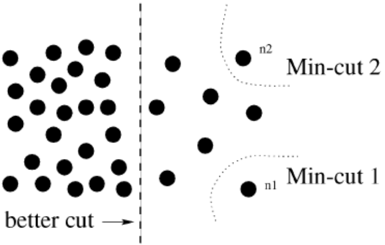

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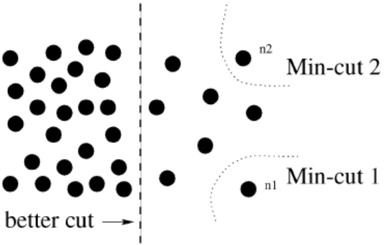

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$$4NCut(A,B) = \frac{[(\mathbf{1}+x) - b(\mathbf{1}-x)]^T (D-W)[(\mathbf{1}+x) - b(\mathbf{1}-x)]}{b\mathbf{1}^T D\mathbf{1}}$$

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Therefore,

$$min_x Ncut(x) = min_y \frac{y^T (D-W) y}{y^T D y}$$

with $y(i) \in \{1, -b\}$ and $y^T D1 = 0$

Continued..

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Let A be a real symmetric matrix. Given x is orthogonal to the $j-1$ smallest eigenvectors $x_1, ..., x_{j-1}$, $\frac{x^T Ax}{x^T x}$ is minimized by the next smallest eigenvector x_j and its minimum value is corresponding λ_j.

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Thus the second smallest eigenvector of the eigensystem gives us the real valued solution to our normalized cut problem

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Running time $O(mn) + O(mM(n))$
m - maximum matrix-vector computations
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W is sparse $\rightarrow$ A is sparse $\rightarrow$ **matrix vector computation is** $O(n)$

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Approach for stabilizing:
Ignore eigenvectors having smoothly varying eigenvector values

Section

Implementation

Images : Matrices of pixel values (Channels)

- Range of pixel values: 0 - 255
- Black and white Images - 1 channel
- Colored images - 3 channels(RGB)

Scikit-image : Image processing library in python

- Has algorithms for image segmentation(including Ncut)

Ncut function in scikit-image library -

- Based on the paper we are discussing
- Creates a Region Adjacency Graph (RAG)
- Recursively performs a Normalized Cut on the RAG

Implementation

For our implementation, we use a 3468 x 4624 pixels image.

Figure: Original image

Implementation

We then tried segmentation using different number of segments, the output for which is as follows -

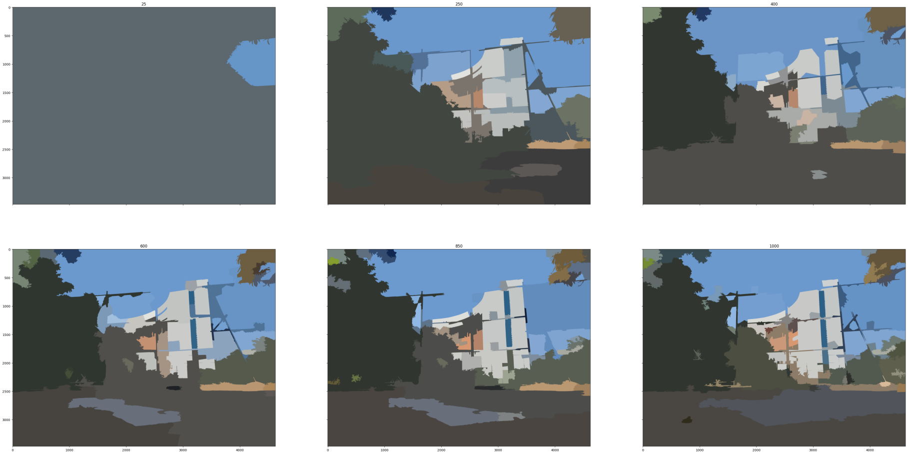


Figure: Segmented images

Implementation

Comparing the original image and a segmented image -

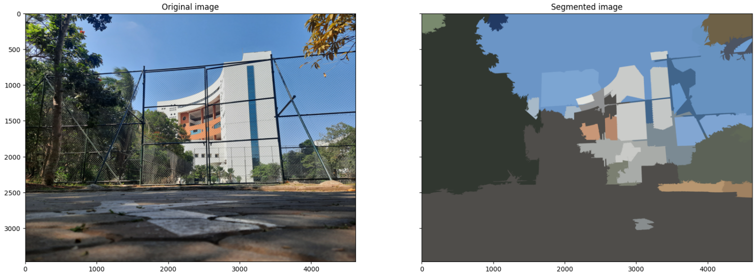


Figure: Comparision

The full code for the image segmentation performed above can be found here - `https://github.com/Swastikamohapatra/Normalized-cut-Image-segmentation`

Section

1. Introduction
2. Normalized Cut
3. Implementation
4. Comparision
5. Conclusion

Comparision

The normalized cut formulation has a certain resemblance to the average cut, as well as the average association formulation. All three of these algorithms can be reduced to solving certain eigenvalue systems.

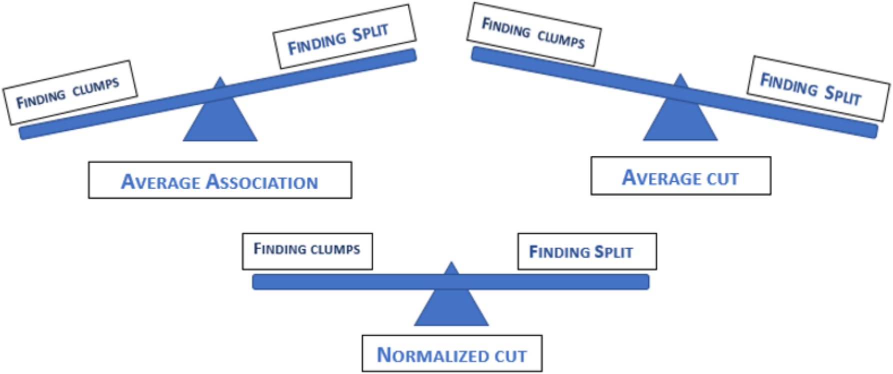


Figure: Comparision

Illustration

To illustrate these above said points, we consider a set of randomly distributed data in 1 Dimension. The 1 Dimensional data is made up of two subsets of points:-

- The first 20 points are randomly distributed from 0 to 0.5.
- The remaining 12 points are randomly distributed from 0.65 to 1

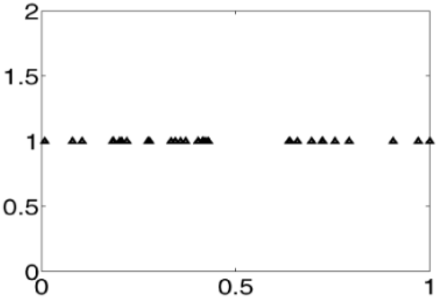


Figure: 1D Data set

Graph

Making a Weighted Graph out of the data set:

- Node = Data Points
- weight = Inversely Proportional to the distance between two Nodes
- Three Monotonically decreasing Weight function with different rate of fall-off:
  - $w(x) = e^{(-d(x)/-0.1)^2}$
  - $w(x) = 1 - d(x)$
  - $w(x) = e^{(-d(x)/-0.2)}$

  Where $d(x)$ is a distance function

First Function

First Function: $w(x) = e^{(-d(x)/-0.1)^2}$

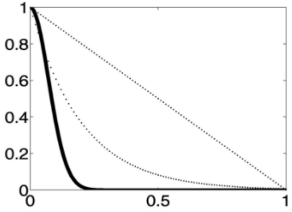


Figure: Weight function

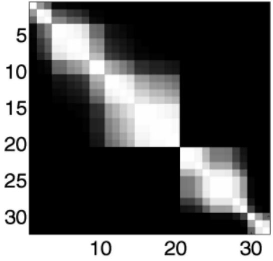
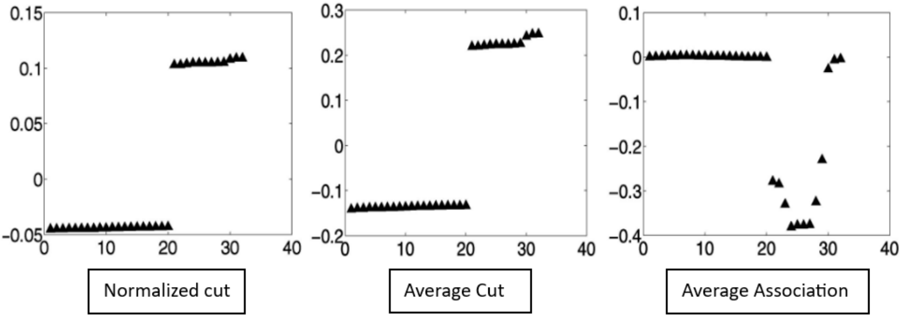


Figure: Graph weight matrix

Note:-

- This function has the fastest decreasing rate among the three.
- Weight is represented by brightness(i.e. higher the weight higher the brightness and vice versa.
- With this weight function, only close-by points are connected.

First Function



Findings/Observation:-

- Using the second extreme eigenvector, both Normalized and Average Cut partitioned the data point into two clusters which is the true situation
- Average Association fails to do the right clustering. It partitioned the data point into isolated small clusters because of its bias towards finding "tight" clusters.

Second Function

Second Function: $w(x) = 1 - d(x)$

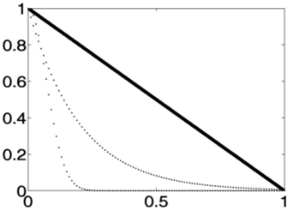


Figure: Weight function

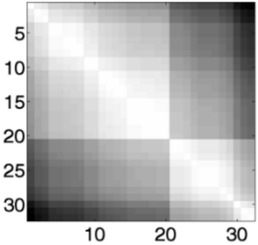
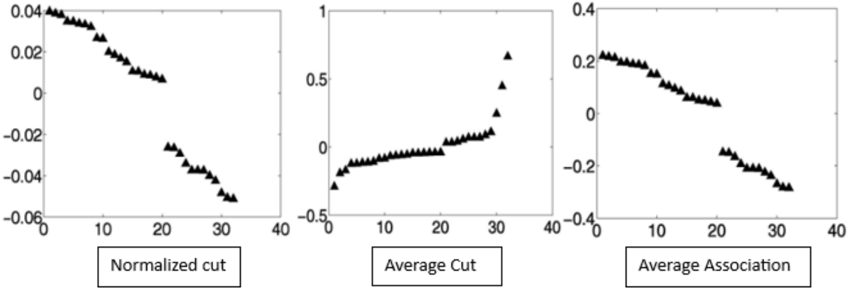


Figure: Graph weight matrix

Note:-

- This function has the slowest decreasing rate among the three.
- With this weight function, most points have non trivial connection to the rest.

Second Function



Findings/Observation:-

- Normalized Cut gives the right partition.
- Average Association also gives the right partition because it easily finds the two the two tight cluster by eliminating few edges with heavy weight across the two clusters.
- Average cut fails because the cluster on the right has less within-group similarity comparision with the cluster on the left. Thus average has trouble deciding on where to cut.

Third Function

Third Function: $w(x) = e^{(-d(x)/-0.2)}$

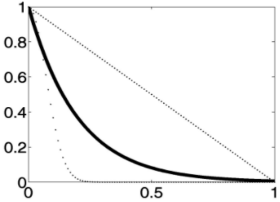


Figure: Weight function

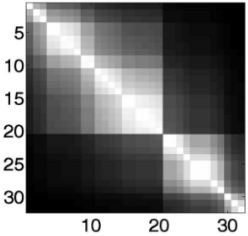
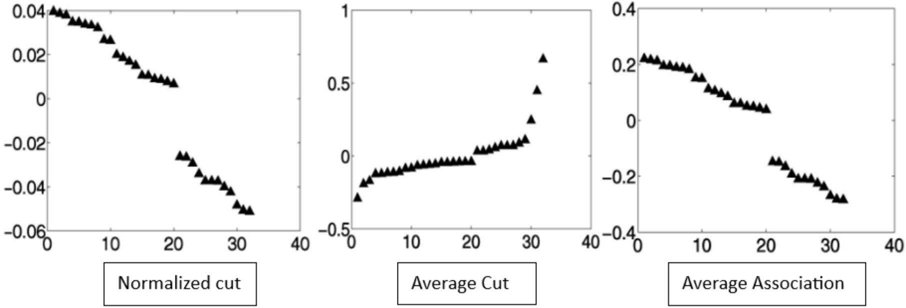


Figure: Graph weight matrix

Note:-

- This function has the moderate decreasing rate among the three.
- with this weight function, the nearby-point connections are balanced against far-away point connections. .

Third Function



Findings/Observation:-

- Normalized cut produces a more clearer solution than the other two because it somehow balance the goal of “Clustering” and “Segmentation”.

Hence, Normalized cut performs well for all these three situation.

Section

Conclusion

- Finally, we got a grouping algorithm that focuses on perceptual grouping and aims to extract the global impression of a picture.
- Minimizing Normalized Cut(unbiased measure of disassociation between subgroups of a graph) leads to directly maximizing the Normalized Association(unbiased measure for total association within the subgroups).
  $Ncut(A, B) = 2 - Nassoc(A, B)$
- Converting the problem of computing the minimum Normalized cut into a problem of solving a generalized eigenvalue system, makes the algorithm more efficient.

Thank You